

The Average Dimension of the Hermitian Hull of Constacyclic Codes over Finite Fields

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Abstract

The hulls of linear and cyclic codes have been extensively studied due to their wide applications. The dimensions and average dimension of the Euclidean hull of linear and cyclic codes have been well-studied. In this paper, the average dimension of the Hermitian hull of constacyclic codes of length n over a finite field \mathbb{F}_{q^2} is determined together with some upper and lower bounds. It turns out that either the average dimension of the Hermitian hull of constacyclic codes of length n over \mathbb{F}_{q^2} is zero or it grows the same rate as n . Comparison to the average dimension of the Euclidean hull of cyclic codes is discussed as well.

Keywords: average dimension, constacyclic codes, Hermitian hull, polynomials

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1. INTRODUCTION

The (Euclidean) hull of a linear code has been introduced to classify finite projective planes in [1]. It is defined to be the intersection of a linear code and its Euclidean dual. Later, it turns out that the hulls of linear codes play a vital role in determining the complexity of algorithms for checking permutation equivalence of two linear codes in [8, 13, 14]. Subsequently, it has been shown that the hull is an indicator for the complexity of algorithms for computing the automorphism group of a linear code in [5, 12]. Precisely, most of the algorithms do not work if the size of the hull is large. Recently, the hulls of linear codes have been applied

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in constructing good entanglement-assisted quantum error correcting codes in [4]. Due to these wide applications, the hulls of linear codes and their properties have been extensively studied. The number of linear codes of length n over \mathbb{F}_q whose hulls have a common dimension and the average dimension of the hull of linear codes were studied in [11]. It has been shown that the average dimension of the hull of linear codes is asymptotically a positive constant dependent of q .

Constacyclic codes constitute an important class of linear codes due to their nice algebraic structures and various applications in engineering [2] and [3]. Especially, this family of codes contains a class of well-studied cyclic codes. In [9], the number of cyclic codes of length n over \mathbb{F}_q having hull of a fixed dimension has been determined together with the dimensions of the hulls of cyclic codes of length n over \mathbb{F}_q . The average dimension of the hull of cyclic codes with respect to the Euclidean and Hermitian inner products have been investigated in [15] and [7], respectively. It has been shown that either the average dimension of the hull of such codes is zero or it grows at the same rate with n . In [10], the dimensions of the Hermitian hulls of constacyclic codes of length n over \mathbb{F}_{q^2} have been determined. However, in the literature, the average dimension of the Hermitian hull of constacyclic codes has not been studied. Therefore, it is of natural interest to study the average dimension of the Hermitian hull of constacyclic codes.

In this paper, we focus on the average dimension of the Hermitian hull of constacyclic codes. Employ the techniques modified from [10] and [15], a general formula for the average dimension of the Hermitian hull of constacyclic codes of length n over \mathbb{F}_{q^2} is determined. Asymptotically, either the average dimension of the Hermitian hull of constacyclic codes is zero or it grows at the same rate with n . This result coincides with the case of the average dimension of the Euclidean hull of cyclic codes. However, there are interesting differences on lower bounds discussed in Section 6.

The paper is organized as follows. In Section 2, some basic knowledge concerning polynomials and codes over finite fields are recalled. A general formula for the average dimension of the Hermitian hull of constacyclic codes is given in Section 3. In Section 4, some number theoretical tools are discussed together with a simplified formula for the average dimension of the Hermitian hull of constacyclic codes. Lower and upper bounds for the average dimension of the Hermitian hull of constacyclic codes are studied in Section 5. The summary and remarks are given in Section 6.

2. PRELIMINARIES

The main focus of this paper is the average dimension of the Hermitian hull of constacyclic codes which is well-defined only over a finite field of square order (see Equation (1)). In this section, some basic properties of codes and polynomials over such finite fields.

For convenience, let p be a prime, q be a p -power integer and let \mathbb{F}_{q^2} denote a finite field of order q^2 and characteristic p . For a given positive integer n , let $\mathbb{F}_{q^2}^n$ denote the \mathbb{F}_{q^2} -vector space of all vectors of length n over \mathbb{F}_{q^2} . For $0 \leq k \leq n$, a *linear code* of length n and dimension k over \mathbb{F}_{q^2} is defined to be a k -dimensional subspace of the \mathbb{F}_{q^2} -vector space $\mathbb{F}_{q^2}^n$. The *Hermitian dual* of a linear code C is defined to be the set

$$C^{\perp_H} := \left\{ \mathbf{u} \in \mathbb{F}_{q^2}^n \mid \mathbf{u} \cdot \mathbf{v}^q = 0 \text{ for all } \mathbf{v} \in C \right\},$$

where $\mathbf{u} \cdot \mathbf{v}^q := \sum_{i=0}^{n-1} u_i \cdot v_i^q$ for all $\mathbf{u} = (u_0, u_1, \dots, u_{n-1})$ and $\mathbf{v} = (v_0, v_1, \dots, v_{n-1})$ in $\mathbb{F}_{q^2}^n$. The *Hermitian hull* of a linear code C is defined to be

$$\text{Hull}_H(C) := C \cap C^{\perp_H}. \quad (1)$$

For a fixed nonzero element λ in \mathbb{F}_{q^2} , a linear code of length n over \mathbb{F}_{q^2} is said to be *constacyclic*, or specifically, *λ -constacyclic* if $(\lambda c_{n-1}, c_1, \dots, c_{n-2}) \in C$ whenever $(c_0, c_1, \dots, c_{n-1}) \in C$. Every λ -constacyclic code C of length n over \mathbb{F}_{q^2} can be identified by an ideal in the principal ideal ring $\mathbb{F}_{q^2}[x]/\langle x^n - \lambda \rangle$ uniquely generated by a monic divisor of $x^n - \lambda$. In this case, $g(x)$ is called the *generator polynomial* for C and we have $\dim C = n - \deg(g(x))$.

For each polynomial $f(x) = a_0 + a_1x + \dots + a_kx^k \in \mathbb{F}_{q^2}[x]$ of degree k and $a_0 \neq 0$, the *conjugate-reciprocal polynomial* of $f(x)$ is defined to be $f^\dagger(x) := a_0^{-q} \sum_{i=0}^k a_i^q x^{k-i}$. It is not difficult to see that $(f^\dagger)^\dagger(x) = f(x)$. A polynomial $f(x)$ is said to be *self-conjugate-reciprocal* if $f(x) = f^\dagger(x)$. Otherwise, $f(x)$ and $f^\dagger(x)$ are called a *conjugate-reciprocal polynomial pair*.

Denote by r the order of an element λ in the multiplication group $\mathbb{F}_{q^2}^* := \mathbb{F}_{q^2} \setminus \{0\}$. In [16, Proposition 2.3], it has been shown that the Hermitian dual of a λ -constacyclic code over \mathbb{F}_{q^2} is again λ -constacyclic if and only if $r|(q+1)$. Based on this characterization, we assume that λ is an element in \mathbb{F}_{q^2} of order r such that $r|(q+1)$ throughout the paper.

Let C be a λ -constacyclic code of length n over \mathbb{F}_{q^2} with the generator polynomial $g(x)$ and let $h(x) = \frac{x^n - \lambda}{g(x)}$. Then $h^\dagger(x)$ is a monic divisor of $x^n - \lambda$ and it is the

generator polynomial of C^{\perp_H} (see [16, Lemma 2.1]). By [9, Theorem 1], the hull $\text{Hull}_H(C)$ of C is generated by the polynomial $\text{lcm}(g(x), h^\dagger(x))$.

Let $C(n, \lambda, q^2)$ denote the set of all λ -constacyclic codes of length n over \mathbb{F}_{q^2} . The *average dimension* of the Hermitian hull of λ -constacyclic codes of length n over \mathbb{F}_{q^2} is defined to be

$$E_H(n, \lambda, q^2) := \sum_{C \in C(n, \lambda, q^2)} \frac{\dim \text{Hull}_H(C)}{|C(n, \lambda, q^2)|}.$$

For each positive integer n , it can be written in the form of $n = \bar{n}p^\nu$, where $p \nmid \bar{n}$ and $\nu \geq 0$. Since the map $\alpha \mapsto \alpha^{p^\nu}$ is an automorphism on \mathbb{F}_{q^2} , there exists an element $\Lambda \in \mathbb{F}_{q^2}$ such that $\Lambda^{p^\nu} = \lambda$ and the multiplicative order of Λ is r . Using arguments similar to those in [10, Section 3], up to permutation, there exist nonnegative integers s and t such that

$$x^n - \lambda = (x^{\bar{n}} - \Lambda)^{p^\nu} = \prod_{i=1}^s (g_i(x))^{p^\nu} \prod_{j=1}^t (f_j(x))^{p^\nu} (f_j^\dagger(x))^{p^\nu}, \quad (2)$$

where $f_j(x)$ and $f_j^\dagger(x)$ are a conjugate-reciprocal polynomial pair and $g_i(x)$ is a monic irreducible self-conjugate-reciprocal polynomial for all $1 \leq i \leq s$ and $1 \leq j \leq t$.

Based on the factorization in Equation (2), the generator polynomial of a λ -constacyclic code C of length n over \mathbb{F}_{q^2} can be viewed of the form

$$g(x) = \prod_{i=1}^s g_i(x)^{u_i} \prod_{j=1}^t f_j(x)^{z_j} (f_j^\dagger(x))^{w_j},$$

where $0 \leq u_i, z_j, w_j \leq p^\nu$. It follows that the generator polynomial of C^{\perp_H} is

$$h^\dagger(x) = \prod_{i=1}^s g_i(x)^{p^\nu - u_i} \prod_{j=1}^t f_j(x)^{p^\nu - w_j} (f_j^\dagger(x))^{p^\nu - z_j},$$

and hence, the generator polynomial of $\text{Hull}_H(C)$ is

$$\text{lcm}(g(x), h^\dagger(x)) = \prod_{i=1}^s g_i(x)^{\max\{u_i, p^\nu - u_i\}} \prod_{j=1}^t f_j(x)^{\max\{z_j, p^\nu - w_j\}} (f_j^\dagger(x))^{\max\{w_j, p^\nu - z_j\}}. \quad (3)$$

It follows that

$$\begin{aligned} \dim \text{Hull}_H(C) = n - \sum_{i=1}^s \deg(g_i(x)) (\max\{u_i, p^\nu - u_i\}) \\ - \sum_{j=1}^t \deg(f_j(x)) (\max\{z_j, p^\nu - w_j\} + \max\{w_j, p^\nu - z_j\}) \end{aligned} \quad (4)$$

3. The Average Dimension $E_H(n, \lambda, q^2)$

In this section, a general formula for the average dimension $E_H(n, \lambda, q^2)$ of the Hermitian hull of λ -constacyclic codes of length n over \mathbb{F}_{q^2} is given together with some upper bounds.

Assume that $x^{\bar{n}} - \Lambda$ has the factorization in the form of Equation (2) and let $B_{\bar{n}, \lambda, q^2}^H := \sum_{i=1}^s \deg(g_i(x))$. The formula for the average dimension of the Hermitian hull of constacyclic codes can be determined using the expectation $E(\cdot)$ in Probability Theory as follows.

Lemma 3.1. *Let p be a prime and let ν be a nonnegative integer. Let $0 \leq u, z, w \leq p^\nu$. Then the following statements hold.*

1. $E(\max\{u, p^\nu - u\}) = \frac{3p^\nu + 1}{4} - \frac{\delta_{p^\nu}}{4(p^\nu + 1)}$, where $\delta_{p^\nu} = 1$ if p^ν is even, and $\delta_{p^\nu} = 0$ otherwise.
2. $E(\max\{z, p^\nu - w\}) = \frac{p^\nu(4p^\nu + 5)}{6(p^\nu + 1)}$.

Proof. The statements can be obtained using arguments similar to those in the proof of [15, Theorem 23]. \square

Theorem 3.2. *Let \mathbb{F}_{q^2} be a finite field of order q^2 and characteristic p and let n be a positive integer such that $n = \bar{n}p^\nu$, $p \nmid \bar{n}$ and $\nu \geq 0$. Let λ be an element in \mathbb{F}_{q^2} of order r such that $r \mid (q + 1)$. Then the average dimension of the Hermitian hull of λ -constacyclic codes of length n over \mathbb{F}_{q^2} is*

$$E_H(n, \lambda, q^2) = n \left(\frac{1}{3} - \frac{1}{6(p^\nu + 1)} \right) - B_{\bar{n}, \lambda, q^2}^H \left(\frac{p^\nu + 1}{12} + \frac{2 - 3\delta_{p^\nu}}{12(p^\nu + 1)} \right). \quad (5)$$

Proof. Let Y be the random variable of the dimension $\dim \text{Hull}_H(C)$, where C is chosen randomly from $C(n, \lambda, q^2)$ with uniform probability. By Lemma 3.1,

Equation (4), and arguments similar to those in the proof of [7, Theorem 3.2], we obtain

$$\begin{aligned}
E_H(n, \lambda, q^2) &= E(Y) \\
&= n - \sum_{i=1}^s \deg(g_i(x)) E(\max\{u_i, p^\nu - u_i\}) \\
&\quad - \sum_{j=1}^t \deg(f_j(x)) E(\max\{z_j, p^\nu - w_j\} + \max\{w_j, p^\nu - z_j\}) \\
&= n - \sum_{i=1}^s \deg(g_i(x)) \left(\frac{3p^\nu + 1}{4} - \frac{\delta_{p^\nu}}{4(p^\nu + 1)} \right) \\
&\quad - \sum_{j=1}^t \deg(f_j(x)) \frac{2p^\nu(4p^\nu + 5)}{6(p^\nu + 1)} \\
&= n - \left(\frac{3p^\nu + 1}{4} - \frac{\delta_{p^\nu}}{4(p^\nu + 1)} \right) B_{\bar{n}, \lambda, q^2}^H - \frac{p^\nu(4p^\nu + 5)}{6(p^\nu + 1)} (\bar{n} - B_{\bar{n}, \lambda, q^2}^H) \\
&= n \left(\frac{1}{3} - \frac{1}{6(p^\nu + 1)} \right) - B_{\bar{n}, \lambda, q^2}^H \left(\frac{p^\nu + 1}{12} + \frac{2 - 3\delta_{p^\nu}}{12(p^\nu + 1)} \right).
\end{aligned}$$

The proof is therefore completed. \square

The following corollary is straightforward from Theorem 3.2.

Corollary 3.3. *Assume the notations as in Theorem 3.2. Then the following statements hold.*

1. $E_H(n, \lambda, q^2) < \frac{n}{3}$.
2. $E_H(\bar{n}, \lambda, q^2) = \frac{\bar{n} - B_{\bar{n}, \lambda, q^2}^H}{4}$.
3. $E_H(\bar{n}, \lambda, q^2) < \frac{\bar{n}}{4}$.

4. Properties of $B_{\bar{n}, \lambda, q^2}^H$

In this section, some properties of $B_{\bar{n}, \lambda, q^2}^H$ are given as well as their applications in determining a simplified formula for $E_H(n, \lambda, q^2)$.

Let $M_q := \{\ell \geq 1 | \ell \text{ divides } q^i + 1 \text{ for some odd positive integer } i\}$ and let

$$\chi := \left\{ j \geq 1 \mid j | \bar{n}r \text{ and } \gcd\left(\frac{\bar{n}r}{j}, r\right) = 1 \right\}. \quad (6)$$

For each positive integer j such that $\gcd(j, q) = 1$, denote by $\text{ord}_j(q)$ the *multiplicative order* of q modulo j .

The formula for $B_{\bar{n}, \lambda, q^2}^H$ can be simplified using the sets M_q and χ as follows.

Lemma 4.1. *Assume that $x^{\bar{n}} - \Lambda$ is factorized as in Equation (2). Then*

$$B_{\bar{n}, \lambda, q^2}^H = \sum_{j \in \chi \cap M_q} \frac{\phi(j)}{\phi(r)},$$

where ϕ is the Euler's totient function.

Proof. By Equation (2), we have

$$x^{\bar{n}} - \Lambda = \prod_{i=1}^s g_i(x) \prod_{j=1}^t f_j(x) f_j^\dagger(x).$$

From [10, Equation (3.11)], $x^{\bar{n}} - \Lambda$ can be factored as

$$x^{\bar{n}} - \Lambda = \prod_{j \in \chi \cap M_q} \prod_{i=1}^{\gamma(j)} g_{ij}(x) \prod_{j \in \chi \setminus M_q} \prod_{i=1}^{\beta(j)} f_{ij}(x) f_{ij}^\dagger(x),$$

where $\gamma(j) = \frac{\phi(j)}{\phi(r) \text{ord}_j(q^2)}$, $\beta(j) = \frac{\phi(j)}{2\phi(r) \text{ord}_j(q^2)}$, $f_{ij}(x)$ and $f_{ij}^\dagger(x)$ are a monic irreducible conjugate-reciprocal polynomial pair of degree $\text{ord}_j(q^2)$, and $g_{ij}(x)$ is a monic irreducible self-conjugate-reciprocal polynomial of degree $\text{ord}_j(q^2)$.

Altogether, it can be concluded that

$$\prod_{i=1}^s g_i(x) = \prod_{j \in \chi \cap M_q} \prod_{i=1}^{\gamma(j)} g_{ij}(x).$$

Hence,

$$\begin{aligned}
B_{\bar{n}, \lambda, q^2}^H &= \sum_{i=1}^s \deg(g_i(x)) \\
&= \sum_{j \in \chi \cap M_q} \gamma(j) \deg(g_{ij}(x)) \\
&= \sum_{j \in \chi \cap M_q} \frac{\phi(j)}{\phi(r) \text{ord}_j(q^2)} \cdot \text{ord}_j(q^2) \\
&= \sum_{j \in \chi \cap M_q} \frac{\phi(j)}{\phi(r)}
\end{aligned}$$

as desired. \square

Remark 4.2. From Lemma 4.1, we have the following facts.

1. If $\lambda = 1$, then $B_{\bar{n}, 1, q^2}^H$ is always positive since $1 \in \chi \cap M_q$.
2. If $\lambda \neq 1$, then $\chi \cap M_q$ can be empty. In this case, $B_{\bar{n}, \lambda, q^2}^H = 0$. For example, $B_{4, 2, 9}^H = 0$ since $\chi \cap M_3 = \emptyset$.

Recall that λ is an element in \mathbb{F}_{q^2} of order r such that $r \mid (q + 1)$. Assume that the prime factorization of $\gcd(\bar{n}, r)$ is of the form

$$\gcd(\bar{n}, r) = 2^{c_0} p_1^{c_1} \dots p_s^{c_s}$$

for some $s \geq 0$, where p_1, p_2, \dots, p_s are distinct odd primes, $c_0 \geq 0$ and $c_i \geq 1$ for all $1 \leq i \leq s$. Then \bar{n} and r can be factorized in the forms of

$$\bar{n} = 2^{\beta(\bar{n})} p_1^{a_1} \dots p_s^{a_s} \mu \quad \text{and} \quad r = 2^{\beta(r)} p_1^{b_1} \dots p_s^{b_s} \tau, \quad (7)$$

where $\beta(\bar{n}) \geq c_0$ and $\beta(r) \geq c_0$ are integers, μ and τ are odd (not necessarily prime) integers relative prime to p_i for all $1 \leq i \leq s$, a_i and b_i are positive integers.

Based on the factorizations above, the presentation of the set χ can be simplified as follows.

Lemma 4.3. *Let \bar{n} and r be positive integers and let χ be defined as in Equation (6). Then*

$$\chi = \begin{cases} \left\{ r 2^{\beta(\bar{n})} \prod_{i=1}^s p_i^{a_i} k \mid k \mid \mu \right\} & \text{if } r \text{ is even,} \\ \left\{ r \prod_{i=1}^s p_i^{a_i} k \mid k \mid 2^{\beta(\bar{n})} \mu \right\} & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Consider the following 2 cases.

Case 1 r is even. We have

$$\begin{aligned}\chi &= \left\{ j \geq 1 \mid j|\bar{n}r \text{ and } \gcd\left(\frac{\bar{n}r}{j}, r\right) = 1 \right\} \\ &= \left\{ 2^{\beta(\bar{n})+\beta(r)} p_1^{a_1+b_1} \dots p_s^{a_s+b_s} \tau k \mid k|\mu \right\} \\ &= \left\{ 2^{\beta(\bar{n})+\beta(r)} \prod_{i=1}^s p_i^{a_i+b_i} \tau k \mid k|\mu \right\}\end{aligned}\tag{8}$$

$$= \left\{ r 2^{\beta(\bar{n})} \prod_{i=1}^s p_i^{a_i} k \mid k|\mu \right\}.\tag{9}$$

Case 2 r is odd. We have

$$\begin{aligned}\chi &= \left\{ j|\bar{n}r \mid \gcd\left(\frac{\bar{n}r}{j}, r\right) = 1 \right\} \\ &= \left\{ p_1^{a_1+b_1} \dots p_s^{a_s+b_s} \tau k \mid k|2^{\beta(\bar{n})}\mu \right\} \\ &= \left\{ \prod_{i=1}^s p_i^{a_i+b_i} \tau k \mid k|2^{\beta(\bar{n})}\mu \right\}\end{aligned}\tag{10}$$

$$= \left\{ r \prod_{i=1}^s p_i^{a_i} k \mid k|2^{\beta(\bar{n})}\mu \right\}.\tag{11}$$

Combining the two cases, the result follows. \square

For integers $i \geq 0$ and $j \geq 1$, we say that 2^i exactly divides j , denoted by $2^i \parallel j$, if 2^i divides j but 2^{i+1} does not divide j .

From [6, Corollary 3.7] and its proof, we have the following proposition.

Proposition 4.4. Let q be a prime power and let $\ell = 2^\beta \bar{\ell}$ be a positive integer such that $\bar{\ell}$ is odd and $\beta \geq 0$. Let $\gamma \geq 0$ be an integer such that $2^\gamma \parallel (q+1)$. If $\ell \in M_q$, then one of the following statements holds.

1. $\ell = 1$ and q is even.
2. $\ell \in \{1, 2\}$ and q is odd.
3. $\ell > 2$ and one of the following statements holds.
 - (a) $\bar{\ell} = 1$ and $2 \leq \beta \leq \gamma$.

(b) $\bar{\ell} \geq 3$, $2 \parallel \text{ord}_p(q)$ for every prime p dividing $\bar{\ell}$, and $0 \leq \beta \leq \gamma$.

The necessary and sufficient conditions for $B_{\bar{n}, \lambda, q^2}^H$ to be non-zero are given as follows.

Lemma 4.5. *Let $\gamma \geq 0$ be such that $2^\gamma \parallel (q+1)$. Then one of the following statements holds.*

1. *If r is even, then $B_{\bar{n}, \lambda, q^2}^H \neq 0$ if and only if $\beta(\bar{n}) + \beta(r) \leq \gamma$ and $r \in M_q$.*
2. *If r is odd, then $B_{\bar{n}, \lambda, q^2}^H \neq 0$ if and only if $r \in M_q$.*

Proof. From Lemma 4.1, it is not difficult to see that $B_{\bar{n}, \lambda, q^2}^H \neq 0$ equivalent to $\chi \cap M_q \neq \emptyset$.

To prove 1), assume that r is even. Suppose that $B_{\bar{n}, \lambda, q^2}^H \neq 0$. Then $\chi \cap M_q \neq \emptyset$. Let j be an element in $\chi \cap M_q$. By Equations (8) and (9), we have

$$j = 2^{\beta(\bar{n})+\beta(r)} \prod_{i=1}^s p_i^{a_i+b_i} \tau k = r 2^{\beta(\bar{n})} \prod_{i=1}^s p_i^{a_i} k$$

for some $k \mid \mu$. Since $j \in M_q$, we have $\beta(\bar{n}) + \beta(r) \leq \gamma$ and $r \in M_q$ by Proposition 4.4.

Conversely, assume that $\beta(\bar{n}) + \beta(r) \leq \gamma$ and $r \in M_q$. By setting $k = 1$, we have

$$j = 2^{\beta(\bar{n})+\beta(r)} \prod_{i=1}^s p_i^{a_i+b_i} \tau = r 2^{\beta(\bar{n})} \prod_{i=1}^s p_i^{a_i} \in \chi \cap M_q$$

since r is even.

To prove 2), assume that r is odd. Assume that $B_{\bar{n}, \lambda, q^2}^H \neq 0$. Then $\chi \cap M_q \neq \emptyset$. Let j be an element in $\chi \cap M_q$. Then

$$j = \prod_{i=1}^s p_i^{a_i+b_i} \tau k = r \prod_{i=1}^s p_i^{a_i} k$$

for some $k \mid 2^{\beta(\bar{n})} \mu$ by Equations (10) and (11). Since $j \in M_q$ and $r \mid j$, we have $r \in M_q$ by Proposition 4.4.

Conversely, assume that $r \in M_q$. By setting $k = 1$, we have

$$j = \prod_{i=1}^s p_i^{a_i+b_i} \tau = r \prod_{i=1}^s p_i^{a_i} \in \chi \cap M_q$$

since r is odd. □

The next lemma can be deduced from Lemma 4.5.

Corollary 4.6. *Let $\gamma \geq 0$ be such that $2^\gamma \parallel (q + 1)$. Then one of the following statements holds.*

1. *If r is even, then $B_{\bar{n}, \lambda, q^2} = 0$ if and only if $\beta(\bar{n}) + \beta(r) > \gamma$ or $r \notin M_q$.*
2. *If r is odd, then $B_{\bar{n}, \lambda, q^2} = 0$ if and only if $r \notin M_q$.*

In the case where $B_{\bar{n}, \lambda, q^2} = 0$, the average dimension $E_H(n, \lambda, q^2)$ can be simplified from Theorem 3.2 and Corollary 4.6 as follows.

Corollary 4.7. *Let q be a power of a prime p and let $n \geq 1$. Then the following statements holds.*

1. *If r is even, then $E_H(n, \lambda, q^2) = n \left(\frac{1}{3} - \frac{1}{6(p^\gamma + 1)} \right)$ if and only if $\beta(\bar{n}) + \beta(r) > \gamma$ or $r \notin M_q$.*
2. *If r is odd, then $E_H(n, \lambda, q^2) = n \left(\frac{1}{3} - \frac{1}{6(p^\gamma + 1)} \right)$ if and only if $r \notin M_q$.*

Let ℓ be a positive integer relatively prime to q . Let $\ell = 2^\beta p_1^{e_1} \dots p_k^{e_k}$ be the prime factorization of ℓ where $e_i \geq 1$, $\beta \geq 0$ and $k \geq 0$. Let $K' := \{i | p_i \notin M_q\}$ and $K_1 := \{i | p_i \in M_q\}$. Then K' and K_1 form a partition of $\{1, \dots, k\}$. Let $d'(\ell) = \prod_{i \in K'} p_i^{e_i}$ and $d_1(\ell) = \prod_{i \in K_1} p_i^{e_i}$. For convenience, the empty product will be regarded as 1. We therefore have $\ell = 2^{\beta(\ell)} d'(\ell) d_1(\ell)$ and it is called the M_q -factorization of ℓ .

From Corollary 4.6, if $r \notin M_q$, then $B_{\bar{n}, \lambda, q^2}^H = 0$. Next, the expression of $B_{\bar{n}, \lambda, q^2}^H$ when $r \in M_q$ is determined using properties of set M_q and the expression of χ in the following proposition. By Proposition 4.4, r is of the form $r = 2^{\beta(r)} d_1(r)$. Since $r | (q + 1)$, we have $2^{\beta(r)} | (q + 1)$.

Proposition 4.8. Let $\bar{n} = 2^{\beta(\bar{n})} d'(\bar{n}) d_1(\bar{n})$ and $r = 2^{\beta(r)} d_1(r)$ be the M_q -factorizations of \bar{n} and r . Then one of the following statements holds.

1. *If r is even, $\beta(\bar{n}) + \beta(r) \leq \gamma$, and $r \in M_q$, then $B_{\bar{n}, \lambda, q^2}^H = 2^{\beta(\bar{n})} d_1(\bar{n})$.*
2. *If r is odd and $r \in M_q$, then $B_{\bar{n}, \lambda, q^2}^H = 2^{\min\{\gamma, \beta(\bar{n})\}} d_1(\bar{n})$.*

Proof. Assume that r is even, $\beta(\bar{n}) + \beta(r) \leq \gamma$, and $r \in M_q$. We consider the following 2 cases.

Case 1 $\gcd(\bar{n}, r) = 1$.

Case 1.1 $\beta(\bar{n}) = 0$. Then $\bar{n} = d'(\bar{n})d_1(\bar{n})$ and $r = 2^{\beta(r)}d_1(r)$. By Lemma 4.1, it can be concluded that

$$B_{\bar{n}, \lambda, q^2}^H = \frac{1}{\phi(r)} \sum_{j \in \chi \cap M_q} \phi(j) = \frac{1}{\phi(r)} \sum_{k|\bar{n}, kr \in M_q} \phi(kr) = \sum_{k|d_1(\bar{n})} \phi(k) = d_1(\bar{n}).$$

Case 1.2 $\beta(\bar{n}) \neq 0$. Then $\bar{n} = 2^{\beta(\bar{n})}d'(\bar{n})d_1(\bar{n})$ and $r = d_1(r)$. Hence, by Lemma 4.1, we have

$$B_{\bar{n}, \lambda, q^2}^H = \frac{1}{\phi(r)} \sum_{j \in \chi \cap M_q} \phi(j) = \frac{1}{\phi(r)} \sum_{k|\bar{n}, kr \in M_q} \phi(kr) = \sum_{k|2^{\beta(\bar{n})}d_1(\bar{n})} \phi(k) = 2^{\beta(\bar{n})}d_1(\bar{n}).$$

Case 2 $\gcd(\bar{n}, r) \neq 1$. Recall that $\bar{n} = 2^{\beta(\bar{n})} \prod_{i=1}^s p_i^{a_i} \mu$ and $r = 2^{\beta(r)} \prod_{i=1}^s p_i^{b_i} \tau$ are factorizations of \bar{n} and r as in Equation (7). By Lemma 4.3, we have

$$\begin{aligned} B_{\bar{n}, \lambda, q^2}^H &= \frac{1}{\phi(r)} \sum_{j \in \chi \cap M_q} \phi(j) = \frac{1}{\phi(r)} \sum_{k|\mu, k \in M_q} \phi\left(r 2^{\beta(\bar{n})} \prod_{i=1}^s p_i^{a_i} k\right) \\ &= \frac{1}{\phi(r)} \sum_{k|\mu, k \in M_q} \phi\left(2^{\beta(\bar{n})+\beta(r)} \prod_{i=1}^s p_i^{a_i+b_i} \tau k\right) \\ &= \frac{1}{\phi(r)} \cdot \phi(2^{\beta(\bar{n})+\beta(r)}) \cdot \phi\left(\prod_{i=1}^s p_i^{a_i+b_i}\right) \cdot \phi(\tau) \cdot \sum_{k|\frac{d_1(\bar{n})}{\prod_{i=1}^s p_i^{a_i}}} \phi(k) \\ &= \frac{1}{\phi(r)} \cdot \frac{2^{\beta(\bar{n})+\beta(r)}}{2} \cdot \prod_{i=1}^s p_i^{a_i+b_i} \cdot \left(1 - \frac{1}{p_i}\right) \cdot \phi(\tau) \cdot \frac{d_1(\bar{n})}{\prod_{i=1}^s p_i^{a_i}} \\ &= \frac{1}{\phi(r)} \cdot \frac{2^{\beta(\bar{n})+\beta(r)}}{2} \cdot \prod_{i=1}^s p_i^{b_i} \cdot \left(1 - \frac{1}{p_i}\right) \cdot \phi(\tau) \cdot d_1(\bar{n}) \\ &= \frac{1}{\phi(r)} \cdot 2^{\beta(\bar{n})} \cdot \phi(2^{\beta(r)}) \cdot \phi\left(\prod_{i=1}^s p_i^{b_i}\right) \cdot \phi(\tau) \cdot d_1(\bar{n}) \\ &= 2^{\beta(\bar{n})} d_1(\bar{n}). \end{aligned}$$

Next, assume that r is odd and $r \in M_q$. Then $r = \prod_{i=1}^s p_i^{b_i} \tau = d_1(r)$. By

Lemma 4.3, we have

$$\begin{aligned}
B_{\bar{n}, \lambda, q^2}^H &= \frac{1}{\phi(r)} \sum_{j \in \chi \cap M_q} \phi(j) = \frac{1}{\phi(r)} \sum_{k | 2^{\beta(\bar{n})} \mu, k \in M_q} \phi\left(r \prod_{i=1}^s p_i^{a_i} k\right) \\
&= \frac{1}{\phi(r)} \sum_{k | 2^{\beta(\bar{n})} \mu, k \in M_q} \phi\left(\prod_{i=1}^s p_i^{a_i+b_i} \tau k\right) \\
&= \frac{1}{\phi(r)} \cdot \phi\left(\prod_{i=1}^s p_i^{a_i+b_i}\right) \cdot \phi(\tau) \cdot \sum_{k \mid \frac{\bar{n}}{\prod_{i=1}^s p_i^{a_i}}, k \in M_q} \phi(k) \\
&= \frac{1}{\phi(r)} \cdot \phi\left(\prod_{i=1}^s p_i^{a_i+b_i}\right) \cdot \phi(\tau) \cdot \sum_{k \mid 2^{\min\{\beta(\bar{n}), \gamma\}} \frac{d_1(\bar{n})}{\prod_{i=1}^s p_i^{a_i}}} \phi(k) \\
&= \frac{1}{\phi(r)} \cdot \phi\left(\prod_{i=1}^s p_i^{a_i+b_i}\right) \cdot \phi(\tau) \cdot 2^{\min\{\beta(\bar{n}), \gamma\}} \cdot \frac{d_1(\bar{n})}{\prod_{i=1}^s p_i^{a_i}} \\
&= \frac{1}{\phi(r)} \cdot \prod_{i=1}^s p_i^{a_i} \cdot \phi\left(\prod_{i=1}^s p_i^{b_i}\right) \cdot \phi(\tau) \cdot 2^{\min\{\beta(\bar{n}), \gamma\}} \cdot \frac{d_1(\bar{n})}{\prod_{i=1}^s p_i^{a_i}} \\
&= 2^{\min\{\beta(\bar{n}), \gamma\}} d_1(\bar{n})
\end{aligned}$$

as desired. \square

By Lemma 4.5 and Proposition 4.8, we obtain the following corollary.

Corollary 4.9. *Let $\gamma \geq 0$ be an integer such that $2^\gamma \parallel (q+1)$. Then $B_{\bar{n}, \lambda, q^2} = \bar{n}$ if and only if $r, \bar{n} \in M_q$ and $\beta(\bar{n}) + \beta(r) \leq \gamma$.*

5. Bounds on $E_H(n, \lambda, q^2)$

In this section, we focus on upper and lower bounds of $E_H(n, \lambda, q^2)$. Based on this bounds, it can be concluded that either $E_H(n, \lambda, q^2)$ is 0 or it grows at the same rate with n as n tends to infinity.

Theorem 5.1. *Let q be a prime power and $2^\gamma \parallel (q+1)$. Let $n = \bar{n}p^\nu$, where $p \nmid \bar{n}$ and $\nu \geq 0$. Let $\lambda \in \mathbb{F}_{q^2}^*$ be such that the multiplicative order of λ is r and $r \mid (q+1)$. Then one of the following statements holds.*

1. $E_H(n, \lambda, q^2) = 0$ if and only if $\beta(\bar{n}) + \beta(r) \leq \gamma$ and $r, n \in M_q$.

2. If r is even, then one of the following statements holds.

- (a) $\frac{n}{6} \leq E_H(n, \lambda, q^2) < \frac{n}{3}$ if $\beta(\bar{n}) + \beta(r) \leq \gamma$, $r \in M_q$ and $n \notin M_q$.
- (b) $\frac{n}{4} \leq E_H(n, \lambda, q^2) < \frac{n}{3}$ if $\beta(\bar{n}) + \beta(r) > \gamma$ or $r \notin M_q$.

3. If r is odd, then one of the following statements holds.

- (a) $\frac{n}{8} \leq E_H(n, \lambda, q^2) < \frac{n}{3}$ if $r \in M_q$ and $n \notin M_q$.
- (b) $\frac{n}{4} \leq E_H(n, \lambda, q^2) < \frac{n}{3}$ if $r \notin M_q$.

Proof. From Equation (5), observe that $E_H(n, q^2) = 0$ if and only if

$$\frac{B_{\bar{n}, \lambda, q^2}^H}{\bar{n}} = \frac{4p^{2\nu} + 2p^\nu}{p^{2\nu} + 2p^\nu + 3 - 3\delta_{p^\nu}}.$$

By Lemma 4.1, it is not difficult to see that $\frac{B_{\bar{n}, \lambda, q^2}^H}{\bar{n}} \leq 1$ and $\frac{B_{\bar{n}, \lambda, q^2}^H}{\bar{n}} = 1$ if and only if $r, \bar{n} \in M_q$ and $\beta(\bar{n}) + \beta(r) \leq \gamma$ by Corollary 4.9. On the other hand, we have $\frac{4p^{2\nu} + 2p^\nu}{p^{2\nu} + 2p^\nu + 3 - 3\delta_{p^\nu}} \geq 1$ and $\frac{4p^{2\nu} + 2p^\nu}{p^{2\nu} + 2p^\nu + 3 - 3\delta_{p^\nu}} = 1$ if and only if $p^\nu = 1$. Therefore, $\frac{B_{\bar{n}, \lambda, q^2}^H}{\bar{n}} = \frac{4p^{2\nu} + 2p^\nu}{p^{2\nu} + 2p^\nu + 3 - 3\delta_{p^\nu}}$ if and only if they are 1. We conclude that $\frac{B_{\bar{n}, \lambda, q^2}^H}{\bar{n}} = \frac{4p^{2\nu} + 2p^\nu}{p^{2\nu} + 2p^\nu + 3 - 3\delta_{p^\nu}}$ if and only if $r, \bar{n} \in M_q$, $\beta(\bar{n}) + \beta(r) \leq \gamma$ and $p^\nu = 1$. Equivalently, $r, n \in M_q$ and $\beta(\bar{n}) + \beta(r) \leq \gamma$. Therefore, Statement 1) is proved.

In any cases, we have the upper bound $E_H(n, \lambda, q^2) < \frac{n}{3}$ from Corollary 3.3. Therefore, it remains to prove only the lower bounds.

To prove 2), assume that r is even.

(a): Assume that $\beta(\bar{n}) + \beta(r) \leq \gamma$, $r \in M_q$, and $n \notin M_q$.

Case 1 $\gcd(n, q) \neq 1$. By Corollary 3.3, we have $E_H(\bar{n}, \lambda, q^2) = \frac{\bar{n} - B_{\bar{n}, \lambda, q^2}^H}{4}$. Hence, by Equation (5), $E_H(n, \lambda, q^2)$ can be expressed as

$$\begin{aligned} \frac{E_H(n, \lambda, q^2)}{n} &= \frac{1}{4} - \frac{1}{4p^\nu} + \frac{\delta_{p^\nu}}{4p^\nu(p^\nu + 1)} + \frac{E_H(\bar{n}, \lambda, q^2)}{n} \left(\frac{p^\nu + 1}{3} + \frac{2 - 3\delta_{p^\nu}}{3(p^\nu + 1)} \right) \\ &\geq \frac{1}{4} - \frac{1}{4p^\nu} + \frac{\delta_{p^\nu}}{4p^\nu(p^\nu + 1)}. \end{aligned}$$

It follows that $\frac{E_H(n, q^2)}{n} \geq \frac{1}{6}$ for all $p^\nu \geq 2$. Hence, $E_H(n, q^2) \geq \frac{n}{6}$. **Case 2** $\gcd(n, q) = 1$. Then $n = \bar{n} = 2^{\beta(\bar{n})} d'(\bar{n}) d_1(\bar{n})$. By Proposition 4.8, we have $B_{\bar{n}, \lambda, q^2}^H = 2^{\beta(\bar{n})} d_1(\bar{n})$.

It follows that

$$\frac{B_{\bar{n}, \lambda, q^2}^H}{n} = \frac{2^{\beta(\bar{n})} d_1(\bar{n})}{2^{\beta(\bar{n})} d'(\bar{n}) d_1(\bar{n})} = \frac{1}{d'(\bar{n})},$$

and hence, $B_{\bar{n}, \lambda, q^2}^H = \frac{n}{d'(\bar{n})}$. Therefore,

$$E_H(n, \lambda, q^2) = \frac{n - B_{\bar{n}, \lambda, q^2}^H}{4} = \frac{1}{4} \left(n - \frac{n}{d'(\bar{n})} \right) = \frac{n}{4} \left(1 - \frac{1}{d'(\bar{n})} \right) \geq \frac{\bar{n}}{4} \left(1 - \frac{1}{3} \right) = \frac{n}{6}$$

Altogether, it can be concluded that $\frac{n}{6} \leq E_H(n, \lambda, q^2)$.

(b): Assume that $\beta(\bar{n}) + \beta(r) > \gamma$ or $r \notin M_q$. By Corollary 4.7, we have

$$E_H(n, \lambda, q^2) = n \left(\frac{1}{3} - \frac{1}{6(p^\nu + 1)} \right) \geq n \left(\frac{1}{3} - \frac{1}{6(1 + 1)} \right) = \frac{n}{4}.$$

To prove 3), assume that r is odd.

(a): Assume that $r \in M_q$ and $n \notin M_q$.

Case 1 $\gcd(n, q) \neq 1$. Similar to the prove of Case 1 in 2), we have $E_H(n, q^2) \geq \frac{n}{6}$.

Case 2 $\gcd(n, q) = 1$. Then $B_{\bar{n}, \lambda, q^2}^H = 2^{\min\{\gamma, \beta(\bar{n})\}} d_1(\bar{n})$ by Proposition 4.8. Thus

$$\frac{B_{\bar{n}, \lambda, q^2}^H}{n} = \frac{2^{\min\{\gamma, \beta(\bar{n})\}} d_1(\bar{n})}{2^{\beta(\bar{n})} d'(\bar{n}) d_1(\bar{n})} = \frac{1}{2^{\beta(\bar{n}) - \min\{\gamma, \beta(\bar{n})\}} d'(\bar{n})}$$

Since $n \notin M_q$, we consider the following 2 cases.

Case 2.1 $d'(\bar{n}) = 1$ and $\beta(\bar{n}) > \gamma$. We have $\frac{B_{\bar{n}, \lambda, q^2}^H}{n} = \frac{1}{2^{\beta(\bar{n}) - \gamma}} \leq \frac{1}{2}$.

Case 2.2 $d'(\bar{n}) > 1$. Then

$$\frac{B_{\bar{n}, \lambda, q^2}^H}{n} = \frac{1}{2^{\beta(\bar{n}) - \min\{\gamma, \beta(\bar{n})\}} d'(\bar{n})} \leq \frac{1}{d'(\bar{n})} \leq \frac{1}{3}.$$

From Cases 2.1 and 2.2, we get $B_{\bar{n}, q^2}^H \leq \frac{n}{2}$ which implies that

$$E_H(n, \lambda, q^2) = \frac{n - B_{\bar{n}, \lambda, q^2}^H}{4} \geq \frac{n - \frac{n}{2}}{4} = \frac{n}{8}.$$

Consequently, we have $\frac{n}{8} \leq E_H(n, \lambda, q^2)$.

(b): Assume that $r \notin M_q$. By Corollary 4.7, we have

$$E_H(n, \lambda, q^2) = n \left(\frac{1}{3} - \frac{1}{6(p^\nu + 1)} \right) \geq n \left(\frac{1}{3} - \frac{1}{6(1 + 1)} \right) = \frac{n}{4}.$$

The proof is completed. \square

Remark 5.2. Assume the notations as in Theorem 5.1. If $n \in M_q$, then $\bar{n} \in M_q$ by Proposition 4.4 which implies that $\beta(\bar{n}) \leq \gamma$. In the case where r is odd, we have $\beta(r) = 0$ which implies that $\beta(\bar{n}) + \beta(r) = \beta(\bar{n}) \leq \gamma$. Hence, if r is odd, it can be concluded that $E_H(n, \lambda, q^2) = 0$ if and only if $r, n \in M_q$.

From Theorem 5.1, it can be concluded that either $E_H(n, \lambda, q^2) = 0$ or it grows at the same rate with n as n tends to infinity.

6. Conclusion and Remarks

For an element λ in \mathbb{F}_{q^2} of order r such that $r|(q+1)$, a general formula for the average dimension $E_H(n, \lambda, q^2)$ of the hull of λ -constacyclic codes has been given in Theorem 3.2 and as well as its lower and upper bounds in Theorem 5.1. Asymptotically, either the average dimension of the Hermitian hull of constacyclic codes is zero or it grows at the same rate with n .

Let $E(n, q)$ denote the average dimension of the Euclidean hull of cyclic codes of length n over \mathbb{F}_q and let $N_q := \{\ell \geq 1 | \ell \text{ divides } q^i + 1 \text{ for some positive integer } i\}$. In [15, Theorem 25], it has been shown that $E(n, q) = 0$ if and only if $n \in N_q$, and $\frac{n}{12} \leq E(n, q) < \frac{n}{3}$ otherwise. This means that either the average dimension of the Euclidean hull of cyclic codes is zero or it grows at the same rate with n . This result coincides the result for Hermitian case in this paper. However, there are interesting difference on lower bounds explained in Table 1.

Order of λ	n	LB	UB	Remarks
r is odd and $r (q+1)$.	$r \in M_q$ and $n \in M_q$	0	0	Remark 5.2
	$r \in M_q$ and $n \notin M_q$	$\frac{n}{8}$	$\frac{n}{3}$	Theorem 5.1
	$r \notin M_q$	$\frac{n}{4}$	$\frac{n}{3}$	
r is even and $r (q+1)$.	$\beta(\bar{n}) + \beta(r) \leq \gamma$, $r \in M_q$ and $n \in M_q$	0	0	Theorem 5.1
	$\beta(\bar{n}) + \beta(r) \leq \gamma$, $r \in M_q$ and $n \notin M_q$	$\frac{n}{6}$	$\frac{n}{3}$	
	$\beta(\bar{n}) + \beta(r) > \gamma$ or $r \notin M_q$	$\frac{n}{4}$	$\frac{n}{3}$	

Table 1: The lower and upper bounds for $E_H(n, \lambda, q^2)$

From the summary in Table 1, it would be interesting to study an improvement on the lower and upper bounds for the average dimension of the hull of constacyclic codes with some restricted lengths.

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